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Reciprocity of Fisher's renormalization relations for the critical indices

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Abstract. Fisher's renormalization relations between the 'ideal' critical indices and the 'true' ones are proved to be symmetric. This property is shown to be independent of the sign of the ideal specific heat index.

1. Introduction

It is experimentally known that the critical indices of solid systems deviate significantly from the 'ideal' indices which are theoretically predicted for the corresponding Ising systems. More specifically, the theoretical specific heat diverges (with positive index α_{th}) while the experimental specific heat seems to be lower, and is usually finite, although sharply cusped (with negative index α_{ex}). Fisher (1968) showed that this disagreement may be explained by a 'hidden' variables theory. He also established 'renormalization relations' between the experimental (subscript ex) and the ideal (subscript th) indices:

$$\alpha_{ex} = -\alpha_{th}/(1-\alpha_{th}), \qquad \zeta_{ex} = \zeta_{th}/(1-\alpha_{th}) \tag{1}$$

where ζ is any critical index describing temperature dependence, excluding α. Shortly after this, Imry *et al* (1973) rederived the same relations by a somewhat modified 'constraint' theory. An immediate substitution verifies that

$$\alpha_{th} = -\alpha_{ex}/(1-\alpha_{ex}), \qquad \zeta_{th} = \zeta_{ex}/(1-\alpha_{ex}). \tag{2}$$

Equations (1) and (2) are reciprocal in the sense that interchanging the roles of the theoretical and experimental values in equation (1) gives equation (2) and vice versa. This reciprocity may be regarded merely as an accidental result of the mathematical structure of equation (1), or it may have a somewhat deeper physical meaning. Equation (2) may be understood by supposing the theoretical indices to be derived from renormalization of the experimental ones, although α_{ex} is negative! It was, indeed, recently shown by Dohm (1974) that renormalization is possible even if $\alpha_{th} < 0$ but only for a special value of the constrained variable. The present paper goes one step further. Our derivation, although quite similar in parts to that of Dohm, is presented here because our approach and emphasis are somewhat different. The derivation will be based mainly on the paper of Imry *et al* (1973 to be referred to as IEB).

2. Derivation

Let us assume that the thermodynamic function G, which describes the system, is given near the critical line $T_c(\xi)$ by:

$$G(T, \xi) = G_c(T, \xi) + G_\pm(T, \xi)|T - T_c(\xi)|^{2-\alpha_{th}} \tag{3}$$

where ξ is a non-ordering field which, with the temperature T, characterizes the system (e.g. pressure). As G_c and $G_\pm$ are sufficiently regular functions, the only irregularities in G and its higher derivatives are introduced by the term $|T - T_c(\xi)|^{2-\alpha_{th}}$. Let us impose a constraint on the system:

$$F(T, \xi, x) = \theta. \tag{4}$$

F is a regular function of its variables, T, ξ, and x, the thermodynamic conjugate of ξ; θ is the constraint parameter.

The principal result of IEB is, for our purpose, their equation (22):

$$(1 + A(\theta))(T - T(\theta)) = \mathrm{sgn}(T - T_c(\xi_R^\theta))[|T - T_c(\xi_R^\theta)| + \pi(\theta)S_\pm|T - T_c(\xi_R^\theta)|^{1-\alpha}]. \tag{5}$$

$\xi = \xi_R^\theta(T)$ is the equation of state of the *constrained* system, $T(\theta)$ is the point of intersection of $\xi = \xi_R^\theta(T)$ with the critical line $\xi = \xi_c(T)$ (or its inverse $T = T_c(\xi)$), $S_\pm$ are regular and positive functions of (T, ξ); $A(\theta)$ and $\pi(\theta)$ are quite complicated expressions of the constraint function F and its parameter θ. Although their explicit expressions do not concern us here, we note that they are proportional to each other; more specifically they may both diverge at the same time to infinity (because each contains a factor which is reciprocal to an expression which may vanish). In the constrained system $T(\theta)$ should replace $T_c(\xi)$ as the critical line. In IEB it is assumed that α_{th} is positive. This case was analysed and three possibilities were discussed:

(i) $\pi(\theta) = 0$, hence

$$T - T_c(\xi_R^\theta) \sim T - T(\theta), \tag{6}$$

which means $\alpha_{ex} = \alpha_{th}$ and no renormalization occurs.

(ii) $\pi(\theta) > 0$. Here $|T - T_c(\xi_R^\theta)|^{1-\alpha_{th}}$ is the dominant term in the right-hand side of equation (5) and we get

$$T - T_c(\xi_R^\theta) \sim \mathrm{sgn}(T - T(\theta))|T - T(\theta)|^{1/(1-\alpha_{th})}. \tag{7}$$

By substituting equation (7) into equation (3), relations (1) are readily deduced and renormalization occurs.

(iii) $\pi(\theta) < 0$ does not concern us here because it does not involve renormalization.

Possibility (i) is quite rare because $\pi(\theta)$ vanishes only for very special constraints (e.g. $F(T, \xi, x) = \xi$), and it is the exception rather than the rule (possibilities (ii) and (iii)).

Dohm (1974) has observed that equation (5) is valid even for $\alpha_{th} < 0$ since it can be derived from equations (3) and (4) with no further assumption. In this case $T - T_c(\xi_R^\theta)$ is usually the dominant term in the right-hand side of equation (5) near $T_c(\xi_R^\theta)$, so that the index does not undergo renormalization: $\alpha_{ex} = \alpha_{th}$. But there is still one exception, when $\pi(\theta)$ and $A(\theta)$ tend to infinity at the same time. In this case, $\pi(\theta)S_\pm|T - T_c(\xi_R^\theta)|^{1-\alpha_{th}}$ becomes the dominant term, and we get, as before for possibility (ii):

$$T - T_c(\xi_R^\theta) \sim \mathrm{sgn}(T - T(\theta))|T - T(\theta)|^{1/(1-\alpha_{th})}. \tag{8}$$

Hence one again returns to the renormalization relation (1).

In this derivation there is *a priori* no distinction between theoretical and experimental values. Let us reconsider the case $\alpha_{\text{th}} > 0$ and possibility (ii), when renormalization occurs. The theoretical value α_{th} is obtained when writing the thermodynamic potential as a function of the parameter ξ. But if, instead, we had initially calculated G as a function of θ, we would have deduced another index $\bar{\alpha}_{\text{th}} < 0$ (which eventually becomes $-\alpha_{\text{th}}/(1-\alpha_{\text{th}}) = \alpha_{\text{ex}}$). We could then have regarded the parameter ξ as the value of an external constraint $\xi = \xi(T, \theta, t)$ (where t is the conjugate of θ). The corresponding 'experimental' indices for this constraint are deduced by the renormalization relation (1):

$$\bar{\alpha}_{\text{ex}} = -\bar{\alpha}_{\text{th}}/(1-\bar{\alpha}_{\text{th}}) = \alpha_{\text{th}} > 0 \qquad \bar{\zeta}_{\text{ex}} = \bar{\zeta}_{\text{th}}/(1-\bar{\alpha}_{\text{th}}) = \zeta_{\text{th}}.$$

This establishes the reciprocity claim, i.e. the theoretical indices are derived from the experimental ones according to the same recipe by which the experimental indices are obtained from the theoretical ones.

This result is concluded from two observations which were probably overlooked in the previous discussion of IEB: (*a*) The relations (3) and (5) are valid when $\alpha_{\text{th}} > 0$ as well as when $\alpha_{\text{th}} < 0$, as observed by Dohm (1974). (*b*) The variable ξ is equivalent to any other variable θ, as long as there is a one to one correspondence between them. ξ may therefore be regarded as an external constraint on a system which is initially described by the variable θ (this is indeed tacitly assumed in IEB).

However, let us emphasize again that for $\alpha_{\text{th}} < 0$ the renormalization is the exception, as rare as the non-renormalization for $\alpha_{\text{th}} > 0$, and is realized only for very special constraints.

This means that in most systems under arbitrary constraint the experimental value for α is negative regardless of the sign of the initial theoretical index. But, on the other hand, at almost any point on the critical line it is possible to define a special constraint which renormalizes the critical index to $\alpha > 0$. This may be done by finding the constraint for which $A(\theta) \sim \pi(\theta) \to \infty$.

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